

Systematic Computation of the Least Unstable Periodic Orbits in Chaotic Attractors

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We show that a recently proposed numerical technique for the calculation of unstable periodic orbits in chaotic attractors is capable of finding the least unstable periodic orbits of any given order. This is achieved by introducing a modified dynamical system which has the same set of periodic orbits as the original chaotic system, but with a tuning parameter which is used to stabilize the orbits selectively. This technique is central for calculations using the stability criterion for the truncation of cycle expansions, which provide highly improved convergence of calculations of dynamical averages in generic chaotic attractors. The approach is demonstrated for the Hénon attractor.

Unstable periodic orbits in chaotic attractors provide a useful hierarchical framework for calculations of properties such as Lyapunov exponents, fractal dimensions and entropies of the attractors [1,2]. Periodic orbits have been used to characterize the attractors in a variety of low dimensional dissipative dynamical systems, including model systems such as discrete maps [1,2] as well as experimental time series [3–6]. In chaotic Hamiltonian systems, series expansions over unstable periodic orbits, within the semi-classical approximation, have been used to calculate the quantum energy level density as well as properties of the wave functions [7]. Cycle expansion techniques gave rise to highly improved convergence, particularly for systems in which the symbolic dynamics is well understood and long periodic orbits are well shadowed by short ones [8,9].

The calculation of unstable periodic orbits in chaotic dynamical systems is a difficult computational problem. The difficulty is that in a chaotic system the numerical error grows exponentially with the length of the orbit. Therefore, only short unstable periodic orbits can be calculated using the standard techniques of map iteration. Moreover, even if some orbits of a given order p are calculated, one has no guarantee that *all* orbits of this order have been found. A numerical technique capable of calculating arbitrarily long periodic orbits to any desired accuracy was introduced in Ref. [10] for the Hénon map [11] and later applied to a variety of other dynamical systems [12–14]. Furthermore, this method provides a systematic framework for the calculation of all periodic orbits of any given order p , in which each orbit is identified by a unique binary symbol sequence $\{s_n\}$, $n = 1, \dots, p$. The number of unstable periodic orbits of order p for the Hénon map increases exponentially with p according to $N(p) = 2^{K_0 p}$, where $K_0 \leq 1$ is the topological entropy.

Therefore, the problem of finding *all* the periodic orbits of order p requires resources exponential in p .

Series expansions over periodic orbits used for calculations of dynamical averages are typically ordered according to the orbit length p [1,9,15,14]. Due to slow convergence and the fact that the number of orbits increases exponentially with p , these series often require a huge number of orbits [15,14]. It was recently proposed [16,17] that using the cycle expansion framework for generic dynamical systems one can obtain better convergence by truncating the expansion according to the stability of the orbits [18] rather than their length p . This proposal is particularly useful since stability truncation does not require detailed understanding of the symbolic dynamics, it tends to preserve the shadowing properties and takes into account only the significant orbits of each length [17].

In this Letter we show that a recently proposed technique [19] provides a systematic framework for the calculation of the least unstable periodic orbits of any given order p . The resulting orbits are sorted according to their Lyapunov exponents starting with the least unstable. The technique is highly flexible and can be applied in a straightforward manner to a great variety of discrete as well as continuous dynamical systems of any dimension. Therefore, it opens the way for employing the proposal of Refs. [16,17] for a great variety of dynamical systems.

We will first describe the method. Given a N -dimensional chaotic dynamical system $U : \vec{r}_{i+1} = \vec{f}(\vec{r}_i)$ we generate a set of dynamical systems through the linear transformation:

$$S_k : \vec{r}_{i+1} = \vec{r}_i + \mathbf{\Lambda}_k [\vec{f}(\vec{r}_i) - \vec{r}_i] \quad (1)$$

where $\mathbf{\Lambda}_k$ are invertible $N \times N$ constant matrices which can be cast in the form $\mathbf{\Lambda}_k = \lambda \mathbf{C}_k$ with $0 < \lambda < 1$. The

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matrices \mathbf{C}_k are orthogonal with only one nonvanishing entry (± 1) per row or column. The systems S_k are *equivalent* to the system U in the sense that there is a one to one correspondence between the fixed points of U and those of S_k . The important advantage of the representation of Eq. (1) however is that, using a sufficiently small value for the parameter λ , the fixed points of the transformed systems S_k become stable and therefore can easily be determined by an iterative process. Furthermore, the radius of convergence of this iterative algorithm turns out to be finite. The above procedure can be easily extended to the higher iterates $\tilde{f}^{(p)}(\vec{r})$ of U [by replacing in Eq. (1) \tilde{f} with $\tilde{f}^{(p)}$] allowing us to determine all the order p cycles of U . The parameter λ is a key quantity here. For a given period p , it operates as a filter allowing the selective stabilization of only those unstable periodic orbits, which possess Lyapunov exponents smaller than a critical value. Therefore, starting the search for unstable periodic orbits within a certain period p with a value of $\lambda \cong O(10^{-1})$ and gradually lowering λ we obtain the list of all unstable orbits of order p , starting with the least unstable orbit and sorted with increasing values of their Lyapunov exponents.

Next, we focus on two dimensional systems. Denote the stability matrix of some periodic orbit of order p by M , its matrix elements by $m_{ij}^{(p)}$ where $i, j = 1, 2$, and the stability eigenvalues of M by $\rho_1^{(p)}$ and $\rho_2^{(p)}$. Without loss of generality we assume $|\rho_1^{(p)}| > |\rho_2^{(p)}|$. In Refs. [19] a minimal set of matrices $\{\mathbf{C}_k | k = 1, \dots, 5\}$, which is necessary and sufficient to achieve stabilization for any kind of hyperbolic fixed points was provided. The hyperbolic fixed points with reflection (namely, fixed points for which at least one of the eigenvalues $\rho_1^{(p)}, \rho_2^{(p)}$ is negative while $|\rho_1^{(p)}| > 1$ and $|\rho_2^{(p)}| < 1$) which satisfy $\rho_1^{(p)} < 0$ become stable in the transformed system of Eq. (1) if we use $\mathbf{A}_1 = \lambda \mathbf{C}_1$ with $\mathbf{C}_1 = \mathbf{1}$, where $\mathbf{1}$ is the unit 2×2 matrix. After a little algebra a simple relation between the eigenvalues $\mu_{1,2}^{(p)}$ of the stabilized system S_1 and the eigenvalues $\rho_{1,2}^{(p)}$ in the original system U can be obtained:

$$\mu_{1,2}^{(p)} = 1 - \lambda(1 - \rho_{1,2}^{(p)}). \quad (2)$$

The stability condition $|\mu_{1,2}^{(p)}| < 1$ leads to the critical value $\rho_c = 2/\lambda - 1$ which represents an upper limit for the magnitude of the eigenvalues $\rho_{1,2}^{(p)}$ of the fixed points which are stabilized for a given λ . This means that only those orbits for which $|\rho_1^{(p)}| < \rho_c$ become stable for the corresponding λ . Therefore, varying λ we can selectively extract those periodic orbits which possess stability eigenvalues less or equal to a given threshold value.

The stabilization process for the case of hyperbolic fixed points with reflection and $\rho_1^{(p)} > 0$ or without reflection involves the matrices $\{\mathbf{C}_k | k = 2, \dots, 5\}$ [19]. For these cases no exact monotonic relationship like Eq. (2) can be derived. However, apart from exceptional cases

(see below), a selective stabilization procedure is possible similar to the case of the matrix \mathbf{C}_1 : the overall tendency is again that large values of λ stabilize only the least unstable periodic orbits and with decreasing value of λ we get more and more of the increasingly unstable periodic orbits. To derive this let us consider without loss of generality the case $m_{11}^{(p)} > m_{22}^{(p)}$. The matrix which has to be used for the stabilization process in this case is

$$\mathbf{C}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3)$$

The eigenvalues $\mu_{1,2}^{(p)}$ of the transformed system S_2 expressed in terms of the eigenvalues $\rho_{1,2}^{(p)}$ of the original system are

$$\mu_{1,2}^{(p)} = 1 - \frac{\lambda}{2} \left[t \mp \sqrt{t^2 - 4(\rho_1^{(p)} - 1)(1 - \rho_2^{(p)})} \right] \quad (4)$$

where $t = m_{11}^{(p)} - m_{22}^{(p)}$. Since typically $\rho_1^{(p)} \gg \rho_2^{(p)}$ one can use the approximation $\text{Tr}(M) = (m_{11}^{(p)} + m_{22}^{(p)}) = (\rho_1^{(p)} + \rho_2^{(p)}) \cong \rho_1^{(p)}$. Inserting this in the expressions for $\rho_{1,2}^{(p)}$ and using $[\rho_1^{(p)}]^2 \gg 4\rho_1^{(p)}\rho_2^{(p)}$ a detailed analysis of the possible cases leads to the statement $\rho_1^{(p)} = \alpha m_{11}^{(p)}$ where α is a factor of order unity. As can be seen from $\rho_{1,2}^{(p)}$ the only exception to this situation occurs if $m_{22}^{(p)}/m_{11}^{(p)} = -1 + \epsilon$, where $0 < \epsilon \ll 1$ which is certainly rare in chaotic systems. We then have $t = \beta \rho_1^{(p)}$ with β being a factor of $O(1)$. If the square root in Eq.(4) is real we have $t^2 > 4(\rho_1^{(p)} - 1)(1 - \rho_2^{(p)})$ and it immediately follows that the relevant larger eigenvalue $\mu_1^{(p)}$ obeys $\mu_1^{(p)} = 1 - \lambda\gamma\rho_1^{(p)}/2$ with γ being a factor of $O(1)$. If the square root is imaginary the real part of $\mu_1^{(p)}$, which is responsible for the stabilization, obeys $\text{Re}(\mu_1^{(p)}) = 1 - \lambda\gamma\rho_1^{(p)}/2$. Exceptional cases here are given by $m_{22}^{(p)}/m_{11}^{(p)} = 1 - \epsilon$ where $0 < \epsilon \ll 1$. Although there is no strict monotonic ordering the above arguments clearly demonstrate that a monotonic ordering occurs to a good approximation. In addition we have performed a random matrix simulation for the original stability matrices respecting the above constraints due to hyperbolicity and calculated the distribution of the resulting prefactors γ occurring in the eigenvalues of the transformed system. The results of this analysis confirm the above-obtained conclusions. The cases involving the other matrices $\{\mathbf{C}_k\}$ can be treated analogously.

To demonstrate the power of the above method we now apply it to the Hénon map. To examine the spectrum of Lyapunov exponents we first calculated all the periodic orbits up to $p = 23$ for the Hénon map [11]

$$x_{n+1} = a - x_n^2 + by_n, \quad y_{n+1} = x_n \quad (5)$$

with the parameters $a = 1.4$ and $b = 0.3$, using the method described in [10]. The total number of orbits

obtained is 118,407 (including cyclic permutations and repetitions of lower cycles). For each one of these orbits we calculated the Lyapunov exponent $h = \log(|\rho|)/p$ where ρ (in absolute value) is the largest eigenvalue of the matrix $M = M_p \cdot \dots \cdot M_2 \cdot M_1$, where

$$M_n = \begin{pmatrix} -2x_n & b \\ 1 & 0 \end{pmatrix} \quad (6)$$

are the Jacobian matrices of the map. The Lyapunov exponents of all orbits of order $p = 1, \dots, 23$ as a function of p are shown in Fig. 1. We observe that in this attractor the two fixed points have the largest Lyapunov exponents and are thus the most unstable. The other Lyapunov exponents form a band that becomes denser and broader as p increases. We also observe a small number of orbits with unusually small Lyapunov exponents. Such orbits appear for orders 13, 16, 18 and 20.

To examine the dependence of Lyapunov exponents on the symbol sequence we plot the Lyapunov exponents for all the periodic orbits of order $p = 20$ vs. the sequential number of the orbit from 0 to $2^{20} - 1$ (Fig. 2). The orbits are ordered such that for every two adjacent orbits in the plot the symbol sequences are different in only one bit. This allows us to examine how the Lyapunov exponents change when one bit is switched in a long symbol sequence. To achieve this the symbol sequence $\{s_n\}$ for each orbit is considered as a Gray code sequence [20]. A Gray to binary transformation $\{s_n\} \rightarrow \{s'_n\}$ is then used and the decimal representation of $\{s'_n\}$ is given in the horizontal axis of Fig. 2.

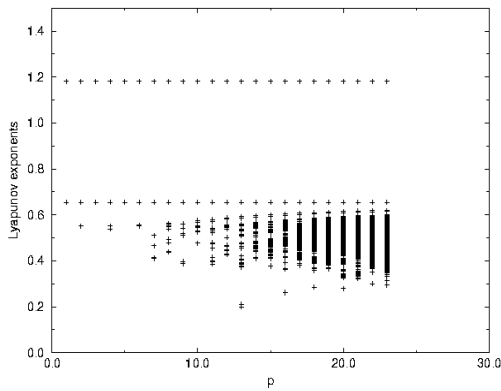


FIG. 1. The Lyapunov exponents of all the periodic orbits of order $p = 1, \dots, 23$ as a function of p for the Hénon attractor at $a = 1.4$ and $b = 0.3$.

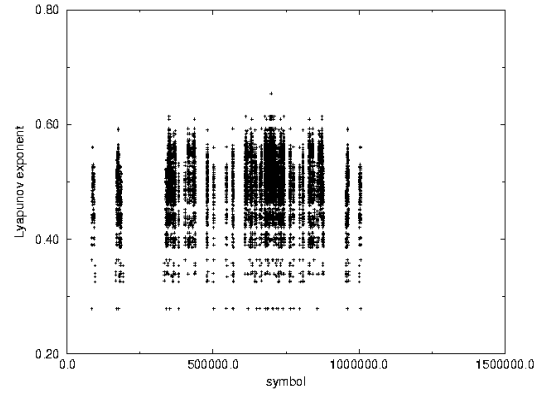


FIG. 2. The Lyapunov exponents for all the orbits of order $p = 20$ in the strange attractor at $a = 1.4$ and $b = 0.3$ as a function of the (decimal representation of the) symbol sequence, converted into the Gray code. The vacant domains in the plot correspond to pruned periodic orbits.

To stabilize the periodic orbits in the Hénon attractor it turns out that one needs only two of the matrices C_k namely $C_1 = 1$ and $C_3 = -C_2$. Using λ -values in the range $(0.05, 0.002)$ and a set of 500 starting points on the attractor we were able to find, for each of the periods $p = 1, \dots, 23$, within a few seconds of computation on a desktop workstation, the two least unstable periodic orbits. The results for $p = 16, \dots, 23$ are presented in Table 1 which shows the period, the (x, y) coordinates of one point of each orbit and its Lyapunov exponent $h_1^{(p)} = (\ln |\rho_1^{(p)}|)/p$.

Period	x-coord.	y-coord.	Lowest Lyap. Exp.
16	1.414441	0.525388	0.261873
16	0.207904	-1.293373	0.259960
17	1.168372	0.719909	0.380900
17	0.956694	0.775439	0.379981
18	-1.255278	1.588681	0.285758
18	-1.256032	1.588953	0.284727
19	0.475407	0.932648	0.365918
19	0.608248	0.683230	0.365689
20	0.999632	0.778785	0.279102
20	0.687520	-0.496536	0.278732
21	1.433765	-0.639919	0.323827
21	1.018516	0.377002	0.323259
22	1.184577	0.641833	0.300455
22	0.641792	0.655127	0.300439
23	1.416001	0.524053	0.295087
23	1.596043	-0.481690	0.294950

TABLE I. The two periodic orbits with the lowest Lyapunov exponents for orders $p = 16, \dots, 23$, for the Hénon map with $a = 1.4$ and $b = 0.3$. The x and y coordinates of one point of each orbit are shown as well as the Lyapunov exponent of the orbit.

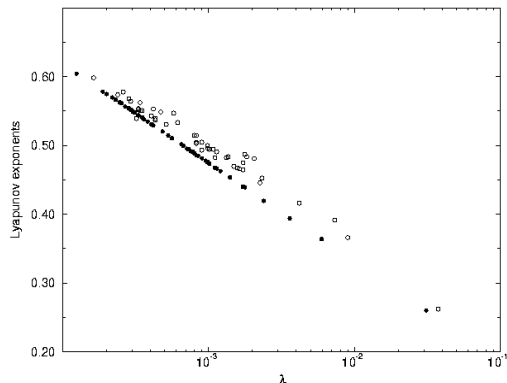


FIG. 3. The Lyapunov exponent of each periodic orbits of period 16 for the Hénon map ($a = 1.4, b = 0.3$) is shown as a function of the critical value of the tuning parameter λ below which the orbit is stabilized. Full circles represent orbits stabilized through C_1 , while open circles represent orbits stabilized through C_3 . For C_1 orbits, the Lyapunov exponents are strictly monotonic vs. λ , while for C_3 orbits only the general trend is monotonic with some deviations.

To demonstrate how the periodic orbits are stabilized as the tuning parameter λ is decreased we chose to present the results for the period $p = 16$. The total number of prime periodic orbits (namely, not including cyclic permutations and repetitions of smaller cycles) for $p = 16$ is 102. For a small value of $\lambda = 10^{-4}$ and 500 starting points on the attractor we get all the 102 orbits stabilized. To examine the stabilization process we start with $\lambda = 0.05$ and gradually lower it. This way, for each periodic orbit we identify the critical value of λ below which the orbit is stabilized. In Fig. 3, we present for all orbits of period $p = 16$ the Lyapunov exponent of each orbit vs. the critical value of λ below which this particular orbit is stabilized. Orbits stabilized with the matrix C_1 are shown in full circles, while orbits stabilized with C_3 are shown in empty circles. We observe that for orbits stabilized with C_1 the Lyapunov exponents increase monotonically as λ is lowered. For orbits stabilized with C_3 monotonicity is not strict, however the general trend is the same. The nearly monotonic tendency of the Lyapunov exponents vs. λ demonstrates the suitability of our approach to determine, with varying λ the set of least unstable periodic orbits within a given period.

We observe that the spectrum of Lyapunov exponents (Fig. 2) exhibits a rough landscape, which resembles the energy spectrum which typically appears in hard minimization problems. Well known problems of this type are finding spin glass ground states [21] and protein folding. The use of combinatorial techniques for these problems is infeasible since the computational resources required are exponential in the input size. However, probabilistic algorithms such as simulated annealing [22] can provide ap-

proximate results, which for many applications are practically sufficient. Our results may provide new insight about hard minimization problems in general. The issue is whether one can artificially destabilize all the metastable states above some externally tuned energy E and this way accelerate the convergence into the low-lying states.

In summary, we have shown that using the method proposed in Refs. [19] one can obtain the periodic orbits of any given order p sorted according to their Lyapunov exponents starting with the least unstable one. The method can be applied to a great variety of discrete as well as continuous dynamical systems of any dimension. Having the periodic orbits sorted in increasing order of their Lyapunov exponents is highly useful in light of the recent proposal [16,17] that in cycle expansion calculations for generic dynamical systems better convergence can be obtained by truncating the expansion according to the stability of the orbits rather than their length p . In particular, stability truncation does not require detailed understanding of the symbolic dynamics, it tends to preserve the shadowing properties and takes into account only the significant orbits of each length, leaving out an exponential number of insignificant orbits.

We thank P. Cvitanović for useful discussions. F.K.D thanks the European Community for financial support. P.S. acknowledges the hospitality of the Max-Planck Institute for Physics of Complex systems.

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- [1] C. Grebogi, E. Ott and J.A. Yorke, Phys. Rev. A **37**, 1711 (1988).
 - [2] P. Cvitanović, G.H. Gunaratne and I. Procaccia, Phys. Rev. A **38**, 1503 (1988).
 - [3] I.B. Schwartz and I. Triandaf, Phys. Rev. A **46**, 7439 (1992).
 - [4] H.D.I. Abarbanel, R. Brown, J.J. Sidorowich and L.Sh. Tsimring, Rev. Mod. Phys. **65**, 1331 (1993).
 - [5] R. Badii, E. Brun, M. Finardi, L. Flepp, R. Holzner, J. Parisi, C. Reyl and J. Simonet, Rev. Mod. Phys. **66**, 1389 (1994).
 - [6] P. So, E. Ott, S.J. Schiff, D.T. Kaplan T. Sauer and C. Grebogi, Phys. Rev. Lett. **76**, 4705 (1996).
 - [7] M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
 - [8] P. Cvitanović, Phys. Rev. Lett. **61**, 2729 (1988).
 - [9] R. Artuso, E. Aurell and P. Cvitanović, Nonlinearity **3**, 325 (1990); Nonlinearity **3**, 361 (1990).
 - [10] O. Biham and W. Wenzel, Phys. Rev. Lett. **63**, 819 (1989).
 - [11] M. Hénon, Commun. Math. Phys. **50**, 69 (1976).
 - [12] O. Biham and W. Wenzel, Phys. Rev. A **42**, 4639 (1990).
 - [13] W. Wenzel, O. Biham and C. Jayaprakash, Phys. Rev. A **43**, 6550 (1991).
 - [14] O. Biham and M. Kvale, Phys. Rev. A **46**, 6334 (1992).

- [15] P. Grassberger, H. Kantz and U. Moenig, J. Phys. A **22**, 5217 (1989).
- [16] C.P. Dettmann and G.P. Morriss, Phys. Rev. Lett. **78**, 4201 (1997).
- [17] C.P. Dettmann and P. Cvitanović, Phys. Rev. E **56**, 6687 (1997).
- [18] P. Dahlqvist and G. Russberg, J. Phys. A **24**, 4763 (1991).
- [19] P. Schmelcher and F.K. Diakonos, Phys. Rev. Lett. **78**, 4733 (1997); Phys. Rev. E **57**, 2739 (1998).
- [20] G.K. Kostopoulos, *Digital Engineering* (John Wiley & Sons, New York, 1975).
- [21] M. Mezard, G. Parisi, and M.A. Virasoro, *Spin Glass Theory and Beyond, World Scientific Lecture Notes in Physics* (World Scientific, Singapore, 1987).
- [22] S. Kirkpatrick, C.D. Gelatt, Jr., and M.P. Vecchi, Science **220**, 671 (1983).